LINEAR MODELS

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Homoskedastic Linear Model

Gauss Markov Assumptions.

- Linearity : $Y = X\beta$
- Strict Exogeneity : $E(\epsilon_i|X) = 0$
  - Unconditional mean of error $E(\epsilon_i) = 0$
  - Cross moment of residuals and regressors is zero, X is orthogonal to ϵ : $E(X_i\epsilon_i) = 0$
- No multicollinearity - $rank(X) = k$
- Spherical error variance : $E(\epsilon_i^2|X) = \sigma^2; E(\epsilon\epsilon'|X) = \sigma^2 I_n$
- $\epsilon|X \sim N(0, \sigma^2 I_n)$
- $(Y_i, X_i) : i = 1, ..., n$ are i.i.d.

This gives us

$$\hat{\beta} = (X'X)^{-1}X'Y$$

$$V(\beta) = \sigma^2(X'X)^{-1}$$

where, under homoskedasticity, $\hat{\sigma^2} = \frac{e'e}{n-k}$, where $e = y - X\beta$

MLE

Density of error:

$$f_{\epsilon_i} = \frac{1}{\sigma\sqrt{2\pi}} exp[-\frac{1}{2}\epsilon_i^2]$$

$$L(\beta, \sigma_\epsilon^2) = \prod_{i=1}^{n} f_{\epsilon_i}(\epsilon_i)$$

Generalised least squares

If covariance matrix of errors is known: $E(\epsilon\epsilon'|X) = \Omega$

$$\hat{\beta}_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}Y$$

$$\mathbb{V}(\hat{\beta}_{GLS}) = (X'\Omega^{-1}X)^{-1}$$

Restrited OLS - optimise: $L(b, \lambda) = (Y - Xb)'(Y - Xb) + 2\lambda(Rb - r)$

Huber-White Sandwich 'Robust' SEs

Under homoskedasticity,

$$E[(\hat{\beta}-\beta)(\hat{\beta}-\beta)'] = (X'X)^{-1}X'E(\epsilon\epsilon')X(X'X)^{-1}$$

which simplifies to $V(\beta) = \sigma^2(X'X)^{-1}$ because of the assumption $E(\epsilon\epsilon') = \sigma^2 I$. If this is not true (i.e. **heteroskedasticity is present**), the variance covariance formula is

$$E[(\hat{\beta}-\beta)(\hat{\beta}-\beta)'] = (X'X)^{-1}X'ee'X(X'X)^{-1} = \sigma_\epsilon^2(X'X)^{-1}X'\Omega X(X'X)^{-1}$$

$V(\hat{\beta}) = Q^{-1}\Omega Q^{-1}$ Where, $Q = \mathbb{E}X_i Xi', \Omega = \mathbb{E}\hat{u_i}^2 X_i X_i'$

Fitted values and residuals

Define 2 matrices that are **positive semidifinite, symmetric,idempotent**:

- $P_x = X(X'X)^{-1}X'$ - Hat Matrix - projector into $span(X)$
- $M_x = I_n - P_x = I_n - X(X'X)^{-1}X'$ - Annihilator Matrix - projector into $span^{\perp}(X)$

Fitted Value: $\hat{Y} = P_x Y$ *Residual*: $e = M_x Y$

Model Fit : R^2, F

R-squared

ESS = Explained Sum of Squares

TSS = Total Sum of Squares

RSS = Residual Sum of Squares

$$R^2 = \frac{ESS}{TSS} = \frac{\sum_{i=1}^n (\hat{Y}-\bar{Y})^2}{\sum_{i=1}^n (Y-\bar{Y})^2} = 1 - \frac{RSS}{TSS} = 1 - \frac{\sum_{i=1}^n (\hat{Y}-Y)^2}{\sum_{i=1}^n (Y-\bar{Y})^2}$$

Adjusted R^2

$$\bar{R}^2 = 1 - \frac{n-1}{n-1}(1-R^2)$$

Mean Squared Error (MSE) $= \mathbb{E}(y - X_i'\hat{\beta})$

$$AIC = ln(\frac{e'e}{n}) + \frac{2k}{n}$$

$$BIC = ln(\frac{e'e}{n}) + \frac{kln(n)}{n}$$

F statistic.

$$\text{F Stat} = (R\hat{\beta}-r)'(s^2R(X'X)^{-1}R')^{-1}(R\hat{\beta}-r)/q$$

$$\text{F Stat} = \frac{(TSS-RSS)/(k-1)}{RSS/(n-k)} = \frac{R^2/(k-1)}{(1-R^2)/(n-k)} \sim F_{k-1,n-k}$$

Wald Statistic.

$$W_n = nh(\hat{\beta}_n)' \left(\frac{\partial h(\hat{\beta}_n)}{\partial \beta'} \hat{V}_n \frac{\partial h(\hat{\beta}_n)'}{\partial \beta} \right) nh(\hat{\beta}_n)$$

reject H_0 if $W_q > \chi^2_{q,1-\alpha} = F/q$

Bonferroni correction - multiple hypothesis correction, J hypotheses : $\tau = \alpha/J$ Holms-Bonferroni : $\alpha/J \dots \alpha/(J-n)$ each step

Instrumental Variables

Exogeneity violated when $E(X_i \epsilon_i) \neq 0$. OLS estimates no longer consistent.

IV requirements:

- $Cov(Z, X) \neq 0$ - Relevance
- $Cov(Z, \epsilon) = 0; Z \perp \epsilon$ - Exogeneity / Exclusion restriction
- Affects Y only through X
- $dim(Z_i) \geq dim(X_i)$

Terminology.

- **First Stage** : Regress X on Z
- **Reduced form** : Regress Y on Z

$$\hat{\beta}_{iv} = (Z'X)^{-1} Z'Y = \frac{Cov(Z, Y)}{Cov(Z, X)} = \frac{\hat{\beta}_{reducedform}}{\hat{\beta}_{firststage}}$$

$$\mathbb{V}(\hat{\beta}_{iv}) = Q_{zx}^{-1} \Omega Q_{xz}^{-1}; \;\; \Omega = \mathbb{E} z_i z_i' u_i^2$$

If $dim(Z_i) > dim(X_i)$ (more instruments than endogenous regressors),

$$\hat{\beta}_{2SLS} = (X' P_z X)^{-1} X' P_z Y = (X'Z(Z'Z)^{-1} Z'X)^{-1} X'Z(Z'Z)^{-1} Z'Y$$

$$\mathbb{V}(\hat{\beta}_{2sls}) = (Q_{xz} Q_{zz}^{-1} Q_{zx})^{-1} Q_{xz} Q_{zz}^{-1} \Omega Q_{zz}^{-1} Q_{zx} (Q_{xz} Q_{zz}^{-1} Q_{zx})^{-1}; \;\; \Omega = \mathbb{E} z_i z_i' u_i^2$$

GMM. If $dim(Z_i) > dim(X_i)$,

$$\hat{\beta}_{gmm}(W) = (X'ZWZ'X)^{-1} X'ZWZ'Y$$

efficient GMM : $\mathbb{V}(\hat{\beta}_{gmm}) = (Q' \Omega^{-1} Q)^{-1}$

Sargan's Over-ID Test. $H_0 : E(Z_i(Y_i - X_i'\beta)) = 0$

$$S = \sum ((Y_i - X_i' \hat{\beta}_{gmm}) Z_i)' (\sum Z_i Z_i')^{-1} \sum ((Y_i - X_i' \hat{\beta}_{gmm}) Z_i) \sim \chi^2_{l-k}$$

Reject if $S > \chi^2_{l-k}$